A non-meta-linguistic theory of truth and implication

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Outline

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Symmetric truth

'... is true' is a predicate that is commonly thought to be symmetric, at least in the following sense:

'φ is true' is always inter-substitutable for φ

where both 'φ is true' and φ belong to the same language.

Object-linguistic, symmetric truth

Why truth as an object-linguistic, symmetric predicate:

- Truth as a logico-linguistic device:
  - Disquotation

    Snow is white if and only if 'Snow is white' is true
  - Blind ascriptions

    everything Socrates said is true
  - Generalizations

    all theorems of Peano Arithmetic are true
- Natural language semantics: truth-conditions for a natural language in that natural language.

Implication as an object-linguistic predicate

Just like '... is true', also '... implies ...' is a predicate, and it is clearly distinct from the propositional connective 'if ... then ...'.

> *Properly, whereas "$\supset$" or "if-then" connects statements, "implies" is a verb which connects names of statements and thus expresses a relation of the named statements.*
>
> *(Quine 1953, pp. 163-164)*

Relationships between implication and truth

Truth and implication are closely connected notions.

- Correct implications preserve truth.
- A sentence is true if and only if it is implied by any sentence.
- An inference is valid if and only if it is necessary that the truth of the premises implies the truth of the conclusion.

⋮

Symmetric implication

How should symmetry be understood for implication?

Intersubstitutivity for implication (Field 2017)

(imp-S) $(\Gamma, \varphi$ implies $\psi)$ if and only Γ implies $(\varphi$ implies $\psi)$

We abbreviate '... implies ...' with imp($\ulcorner\urcorner, \ulcorner\urcorner$)

Naïve rules for symmetric implication (Beall and Murzi 2013)

(imp-I) if $\text{imp}(\ulcorner\Gamma, \varphi\urcorner, \ulcorner\psi\urcorner)$ then $\text{imp}(\ulcorner\Gamma\urcorner, \ulcorner\text{imp}(\ulcorner\varphi\urcorner, \ulcorner\psi\urcorner)\urcorner)$

(imp-E) if $\text{imp}(\ulcorner\Gamma\urcorner, \ulcorner\varphi\urcorner)$ and $\text{imp}(\ulcorner\Delta\urcorner, \ulcorner\text{imp}(\ulcorner\varphi\urcorner, \ulcorner\psi\urcorner)\urcorner)$, then $\text{imp}(\ulcorner\Gamma, \Delta\urcorner, \ulcorner\psi\urcorner)$

Object-linguistic, symmetric implication

The standard motivations for treating truth as an object-linguistic predicate extend to implication:

- Implication as a logico-linguistic device:
  - Blind ascriptions

    what Emmanuel Macron said does not imply anything about Australia

  - Generalizations

    what Emmanuel Macron said implies all theorems of Peano Arithmetic

- Natural language semantics: implication between truth-conditions, incompatibility between truth-conditions, . . .

First aim of this work

In this paper, we provide a theory of symmetric implication and truth as object-linguistic predicates. We provide both a semantic theory and an axiomatic theory.

Outline

Introduction: symmetric, object-linguistic implication and truth

Paradoxes of implication

A semantic theory of symmetric implication and truth

An axiomatic theory of symmetric implication and truth

Implication-Curry (Beall and Murzi 2013)

Symmetric implication yields paradoxes just as symmetric truth.

$(*)$ The sentence labelled with $(*)$ implies "$0 \neq 0$".

1. $(*)$ implies $(*)$ [Reflexivity]
2. $(*)$ implies $\big((*)$ implies $0 \neq 0\big)$ [Reflexivity and def. of $(*)$]
3. $(*)$, $(*)$ imply $0 \neq 0$ [(imp-E): 1, 2]
4. $(*)$ implies $0 \neq 0$ [Contraction: 3]
5. $\varnothing$ implies $\big((*)$ implies $0 \neq 0\big)$ [(imp-I): 4]
6. $\varnothing$ implies $(*)$ [Def. of $(*)$]
7. $0 \neq 0$ [(imp-E): 6, 4]

The Implication-Curry (in this formulation) only employs structural rules.

What is symmetric implication?

- The Implication-Curry and other paradoxes show that symmetric implication is going to be highly non-classical.
- In particular, it will be non-reflexive, non-contractive (or, possibly, non-transitive).
- This clearly separates symmetric implication from the classical conditional (and several non-classical ones).
- What is symmetric implication, then?
- First, two things that are not symmetric implication: logical consequence and derivability.

Implication vs. logical consequence

$$\varphi \vdash \psi$$

$$\Downarrow$$

$$T \vdash \mathsf{imp}(\ulcorner\varphi\urcorner, \ulcorner\psi\urcorner)$$

$$\Downarrow$$

$$T^+ \vdash \mathsf{imp}(\ulcorner\top\urcorner, \ulcorner\mathsf{imp}(\ulcorner\varphi\urcorner, \ulcorner\psi\urcorner)\urcorner)$$

Implication vs. derivability

$$T \vdash \mathsf{imp}(\ulcorner\varphi\urcorner, \ulcorner\psi\urcorner) \ \text{ and } \ T \vdash \varphi$$

$$\Downarrow$$

$$T^{+} \vdash \psi$$

Second aim of this work

In this paper, we also try to suggest some kind of reading for the (necessarily non-classical) notion of symmetric implication.

Outline

Introduction: symmetric, object-linguistic implication and truth

Paradoxes of implication

A semantic theory of symmetric implication and truth

An axiomatic theory of symmetric implication and truth

Plan of the section

- We work in the language of first-order arithmetic, enriched with a binary predicate $\mathsf{imp}(\cdot, \cdot)$. We call this language $\mathcal{L}_{\mathsf{imp}}$.
- We now provide a semantic construction for symmetric implication in $\mathcal{L}_{\mathsf{imp}}$ that generalizes Kripke's (1975) construction for naïve truth (strong Kleene version).

An inductive construction (Nicolai & Rossi 2017)

Let $S \subseteq \omega$, and define the set S^+ as follows. $n \in S^+$ if:

(i) $n \in S$, or

(ii) n is $(\Gamma \Rightarrow s = t, \Delta)$ and s and t have the same value, or

(iii) n is $(\Gamma, s = t \Rightarrow \Delta)$ and s and t have a different value, or

(iv) n is $(\Gamma \Rightarrow \varphi \wedge \psi, \Delta)$ and $(\Gamma \Rightarrow \varphi, \Delta) \in S$ and $(\Gamma \Rightarrow \psi, \Delta) \in S$, or

(v) n is $(\Gamma, \varphi \wedge \psi \Rightarrow \Delta)$ and $(\Gamma, \varphi, \psi \Rightarrow \Delta) \in S$, or

(vi) n is $(\Gamma \Rightarrow \forall x \varphi(x), \Delta)$ and for all $t \in \mathrm{CTer}_{\mathcal{L}_{\mathrm{imp}}}$ $(\Gamma \Rightarrow \varphi(t), \Delta) \in S$, or

(vii) n is $(\Gamma, \forall x \varphi(x) \Rightarrow \Delta)$ and for a $t \in \mathrm{CTer}_{\mathcal{L}_{\mathrm{imp}}}$, $(\Gamma, \varphi(t) \Rightarrow \Delta) \in S$, or

(viii) n is $(\Gamma \Rightarrow \mathrm{imp}(\ulcorner\varphi\urcorner, \ulcorner\psi\urcorner), \Delta)$ and $(\Gamma, \varphi \Rightarrow \psi, \Delta) \in S$, or

(ix) n is $(\Gamma, \mathrm{imp}(\ulcorner\varphi\urcorner, \ulcorner\psi\urcorner) \Rightarrow \Delta)$, $(\Gamma \Rightarrow \varphi, \Delta) \in S$ and $(\Gamma, \psi \Rightarrow \Delta) \in S$.

An inductive construction (cont.)

- We associate a monotone operator $\Psi \ : \mathcal{P}(\omega) \longmapsto \mathcal{P}(\omega)$ to the above definition:

$$\Psi(S) := \{n \in \omega \,|\, \zeta(n, S)\}$$

  where $\zeta(n, S)$ is the disjunction (i) $\vee \ldots \vee$ (ix).
- For every $S \subseteq \omega$, the set

$$S_\Psi := \bigcup_{\alpha \in Ord} \Psi^\alpha(S)$$

  is a fixed point of Ψ.
- We denote with I_Ψ the least fixed point of Ψ

$$\mathsf{I}_\Psi := \bigcup_{\alpha \in Ord} \Psi^\alpha(\varnothing) \subseteq S_\Psi.$$

Some properties of I_Ψ

Proposition

I_Ψ *is consistent* ($\varnothing \Rightarrow \varnothing \notin \mathsf{I}_\Psi$).

Lemma (Weakening)

For every ordinal α, *if* $\Gamma \Rightarrow \Delta$ *is in* I_Ψ^α, *for every* $\Gamma', \Delta' \subseteq \mathsf{Sent}_{\mathcal{L}_{\mathsf{imp}}}$, $\Gamma, \Gamma' \Rightarrow \Delta, \Delta'$ *is in* I_Ψ^α.

Lemma (Contraction)

For every ordinal α, *if* $\Gamma, \varphi, \varphi \Rightarrow \Delta$ *is in* I_Ψ^α, *then* $\Gamma, \varphi \Rightarrow \Delta$ *is in* I_Ψ^α. *Similarly, if* $\Gamma \Rightarrow \psi, \psi, \Delta$ *is in* I_Ψ^α, *then* $\Gamma \Rightarrow \psi, \Delta$ *is in* I_Ψ^α.

Some properties of I_Ψ (cont.)

Lemma (Groundedness)

If $\Gamma \Rightarrow \Delta \in \mathsf{I}^\alpha_\Psi$, then there is a sentence $\varphi \in \Gamma$ s.t. $\varphi \Rightarrow \varnothing \in \mathsf{I}^\alpha_\Psi$, or a sentence $\psi \in \Delta$ s.t. $\varnothing \Rightarrow \psi \in \mathsf{I}^\alpha_\Psi$.

Proof sketch.

Let $\Gamma \Rightarrow \Delta', \forall x \varphi(x) \in \mathsf{I}^{\alpha+1}_\Psi$ be obtained by applying the Ψ-clause for introducing $\forall$ on the right. In I^α_Ψ, we have:

$$(1) \qquad \Gamma \Rightarrow \Delta', \varphi(t_0), \ldots, \Gamma \Rightarrow \Delta', \varphi(t_n), \ldots$$

By IH, for every $\Gamma \Rightarrow \Delta', \varphi(t_i)$ in (1), there is a ψ_i in Γ s.t. $\psi_i \Rightarrow \varnothing \in \mathsf{I}^\alpha_\Psi$, or a χ_i in $\Delta', \varphi(t_i)$ s.t. $\varnothing \Rightarrow \chi_i \in \mathsf{I}^\alpha_\Psi$.
If, for some i, ψ_i or $\chi_i \in \Gamma$ or Δ', we are done. If there is no i s.t. ψ_i or $\chi_i \in \Gamma$ or Δ', by IH $\varnothing \Rightarrow \varphi(t_i) \in \mathsf{I}^\alpha_\Psi$ for all i. Therefore, by an application of the Ψ-clause (ix) $\varnothing \Rightarrow \forall x \varphi(x) \in \mathsf{I}^{\alpha+1}_\Psi$. □

Some properties of I_Ψ (cont.)

Lemma (Inversion)

For every ordinal α, the following holds:

(i) *If* $\Gamma \Rightarrow \varphi \wedge \psi, \Delta \in \mathsf{I}_\Psi^\alpha$, *then* $\Gamma \Rightarrow \varphi, \Delta \in \mathsf{I}_\Psi^\alpha$ *and* $\Gamma \Rightarrow \psi, \Delta \in \mathsf{I}_\Psi^\alpha$.

(ii) *If* $\Gamma, \varphi \wedge \psi \Rightarrow \Delta \in \mathsf{I}_\Psi^\alpha$, *then* $\Gamma, \varphi, \psi \Rightarrow \Delta \in \mathsf{I}_\Psi^\alpha$.

(iii) *If* $\Gamma \Rightarrow \forall x \varphi(x), \Delta \in \mathsf{I}_\Psi^\alpha$, *then for all* $t \in \mathsf{Cter}_{\mathcal{L}_{\mathsf{imp}}}$*:* $\Gamma \Rightarrow \varphi(t), \Delta \in \mathsf{I}_\Psi^\alpha$.

(iv) *If* $\Gamma, \forall x \varphi(x) \Rightarrow \Delta \in \mathsf{I}_\Psi^\alpha$, *then for some* $t \in \mathsf{Cter}_{\mathcal{L}_{\mathsf{imp}}}$*:* $\Gamma, \varphi(t) \Rightarrow \Delta \in \mathsf{I}_\Psi^\alpha$.

(v) *If* $\Gamma \Rightarrow \mathsf{imp}(\ulcorner \varphi \urcorner, \ulcorner \psi \urcorner), \Delta \in \mathsf{I}_\Psi^\alpha$, *then* $\Gamma, \varphi \Rightarrow \psi, \Delta \in \mathsf{I}_\Psi^\alpha$.

(vi) *If* $\Gamma, \mathsf{imp}(\ulcorner \varphi \urcorner, \ulcorner \psi \urcorner) \Rightarrow \Delta \in \mathsf{I}_\Psi^\alpha$, *then* $\Gamma \Rightarrow \varphi, \Delta \in \mathsf{I}_\Psi^\alpha$ *and* $\Gamma, \psi \Rightarrow \Delta \in \mathsf{I}_\Psi^\alpha$.

Some properties of I_Ψ (cont.)

Proposition (Closure under cut)

For every α, if $\Gamma \Rightarrow \Delta, \varphi$ and $\varphi, \Gamma \Rightarrow \Delta$ are in I_Ψ^α, then also $\Gamma \Rightarrow \Delta$ is in I_Ψ^α.

Structure of the proof (Cut-elimination-like).

Let $\Gamma \Rightarrow \Delta, \varphi$ and $\varphi, \Gamma \Rightarrow \Delta$ be in $\mathsf{I}_\Psi^{\alpha+1}$.

(a) $\Gamma \Rightarrow \Delta, \varphi$ and $\varphi, \Gamma \Rightarrow \Delta$ are obtained via a Ψ-clause that introduces φ. [Easy: by induction via weakening]

(b) Only one of $\Gamma \Rightarrow \Delta, \varphi$ and $\varphi, \Gamma \Rightarrow \Delta$ is obtained via a Ψ-clause that introduces φ. [By induction via weakening and inversion]

(c) Neither $\Gamma \Rightarrow \Delta, \varphi$ nor $\varphi, \Gamma \Rightarrow \Delta$ is obtained via a Ψ-clause that introduces φ. [By induction via weakening, inversion, and groundedness]

□

Structural principles that I_Ψ does not have: reflexivity

Lemma

I_Ψ *cannot contain all the instances of*

(Ref) $$\varphi \Rightarrow \varphi$$

for φ an arbitrary $\mathcal{L}_{\mathsf{imp}}$-sentence.

Lemma

I_Ψ *contains all the instances of*

(Ref) $$\varphi \Rightarrow \varphi$$

for φ an $\mathcal{L}$-grounded sentence.

Symmetric implication

Several principles for symmetric implication are recovered in I_Ψ.

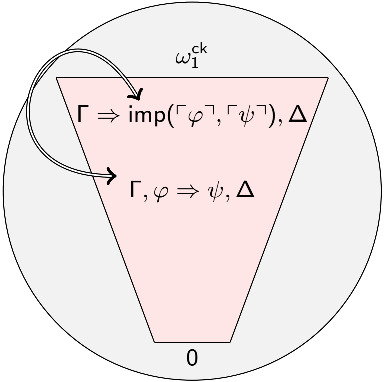
Lemma

For every $\varphi, \psi \in \mathcal{L}_{\mathsf{imp}}$, *and* $\Gamma_0, \Gamma_1, \Delta_0, \Delta_1 \subseteq \mathsf{Sent}_{\mathcal{L}_{\mathsf{imp}}}$:

(imp-S) $\Gamma, \varphi \Rightarrow \psi, \Delta \in \mathsf{I}_\Psi$ *iff* $\Gamma \Rightarrow \mathsf{imp}(\ulcorner\varphi\urcorner, \ulcorner\psi\urcorner), \Delta \in \mathsf{I}_\Psi$.

(imp-E) *if* $\Gamma_0 \Rightarrow \varphi, \Delta_0 \in \mathsf{I}_\Psi$ *and* $\Gamma_1 \Rightarrow \mathsf{imp}(\ulcorner\varphi\urcorner, \ulcorner\psi\urcorner), \Delta_1 \in \mathsf{I}_\Psi$,
then $\Gamma_0, \Gamma_1 \Rightarrow \psi, \Delta_0, \Delta_1 \in \mathsf{I}_\Psi$.

imp-S (and also imp-I)

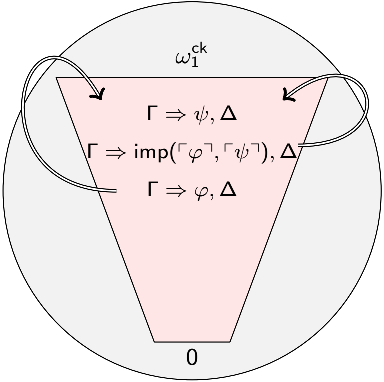


$$\Gamma, \varphi \Rightarrow \psi, \Delta \in \mathsf{I}_{\psi}$$

if and only if

$$\Gamma \Rightarrow \mathrm{imp}(\ulcorner\varphi\urcorner, \ulcorner\psi\urcorner), \Delta \in \mathsf{I}_{\psi}.$$

imp-E



If $\Gamma \Rightarrow \varphi, \Delta \in \mathsf{I}_\Psi$ and

$\Gamma \Rightarrow \mathsf{imp}(\ulcorner\varphi\urcorner, \ulcorner\psi\urcorner), \Delta \in \mathsf{I}_\Psi$,

then $\Gamma \Rightarrow \psi, \Delta \in \mathsf{I}_\Psi$.

Implication principles that I_Ψ does not have

Lemma

I_Ψ *cannot contain all the instances of*

(imp-E*) $$\varphi, \mathsf{imp}(\ulcorner\varphi\urcorner, \ulcorner\psi\urcorner) \Rightarrow \psi$$

for φ, ψ arbitrary $\mathcal{L}_{\mathsf{imp}}$-sentences.

Lemma

I_Ψ *contains all the instances of*

(imp-E*) $$\varphi, \mathsf{imp}(\ulcorner\varphi\urcorner, \ulcorner\psi\urcorner) \Rightarrow \psi$$

for φ, ψ $\mathcal{L}$-grounded sentences.

Ψ and Kripke's original construction (Kripke 1975)

- Let $\neg\varphi := \text{imp}(\ulcorner\varphi\urcorner, \ulcorner\bot\urcorner)$, $\text{Tr}(\ulcorner\varphi\urcorner) := \text{imp}(\ulcorner\top\urcorner, \ulcorner\varphi\urcorner)$.
- Let I_K be the least Kripke fixed point (strong Kleene) for $\mathcal{L}_{\text{imp}}$.

Lemma

For every $\varphi \in \mathcal{L}_{\text{imp}}$:

if φ is in the extension of Tr *in I_K, then $\varnothing \Rightarrow \varphi$ is in I_Ψ;*

if φ is in the anti-extension of T *in I_K, then $\varphi \Rightarrow \varnothing$ is in I_Ψ.*

The opposite direction does not hold.

Corollary

I_Ψ *is closed under the naïve rules for truth:*

$\Gamma \Rightarrow \varphi, \Delta \in \text{I}_\Psi$ *if and only if* $\Gamma \Rightarrow Tr(\ulcorner\varphi\urcorner), \Delta \in \text{I}_\Psi$.

$\Gamma, \varphi \Rightarrow \Delta \in \text{I}_\Psi$ *if and only if* $\Gamma, Tr(\ulcorner\varphi\urcorner) \Rightarrow \Delta \in \text{I}_\Psi$.

Non-minimal fixed points and extensions

- I_Ψ is closed under all the structural meta-inferences. This is not necessarily so for non-minimal fixed points.
- E.g. $\{\varnothing \Rightarrow \mu\}_\Psi$, where $\mu = \mathrm{imp}(\ulcorner\mu\urcorner, \ulcorner\mu\urcorner)$ is not closed under weakening.
- Let Ψ^+ be the monotone operator that results from Ψ by adding an explicit clause for weakening:
  - (x) n is $(\Gamma, \Gamma' \Rightarrow \Delta', \Delta)$, and $(\Gamma \Rightarrow \Delta) \in S$.

Lemma

1. $\mathsf{I}_\Psi = \mathsf{I}_{\Psi^+}$.
2. *For every $S \subseteq \omega$, S_Ψ is consistent iff S_{Ψ^+} is consistent.*

Non-minimal fixed points and extensions (cont.)

Ψ^+ guarantees closure under all the structural and implication meta-rules:

Proposition

For every $S \subseteq \omega$, $\varphi, \psi \in \mathcal{L}_{\mathsf{imp}}$, *and* $\Gamma, \Gamma_0, \Delta, \Delta_0 \subseteq \mathsf{Sent}_{\mathcal{L}_{\mathsf{imp}}}$:

(L-Wkn) *If* $\Gamma \Rightarrow \Delta \in S_{\Psi^+}$, *then* $\Gamma, \varphi \Rightarrow \Delta \in S_{\Psi^+}$.

(R-Wkn) *If* $\Gamma \Rightarrow \Delta \in S_{\Psi^+}$, *then* $\Gamma \Rightarrow \varphi, \Delta \in S_{\Psi^+}$.

(L-Ctr) *If* $\Gamma, \varphi, \varphi \Rightarrow \Delta \in S_{\Psi^+}$, *then* $\Gamma, \varphi \Rightarrow \Delta \in S_{\Psi^+}$.

(R-Ctr) *If* $\Gamma \Rightarrow \varphi, \varphi, \Delta \in S_{\Psi^+}$, *then* $\Gamma \Rightarrow \varphi, \Delta \in S_{\Psi^+}$.

(Cut) *If* $\Gamma \Rightarrow \varphi, \Delta \in S_{\Psi^+}$ *and* $\Gamma, \varphi \Rightarrow \Delta \in S_{\Psi^+}$, *then* $\Gamma \Rightarrow \Delta \in S_{\Psi^+}$.

(imp-S) $\Gamma, \varphi \Rightarrow \psi, \Delta \in \mathsf{I}_{\Psi}$ *iff* $\Gamma \Rightarrow \mathsf{imp}(\ulcorner \varphi \urcorner, \ulcorner \psi \urcorner), \Delta \in \mathsf{I}_{\Psi}$.

(imp-E) *if* $\Gamma \Rightarrow \varphi, \Delta \in \mathsf{I}_{\Psi}$ *and* $\Gamma_0 \Rightarrow \mathsf{imp}(\ulcorner \varphi \urcorner, \ulcorner \psi \urcorner), \Delta_0 \in \mathsf{I}_{\Psi}$, *then* $\Gamma, \Gamma_0 \Rightarrow \psi, \Delta, \Delta_0 \in \mathsf{I}_{\Psi}$.

Models of $\mathcal{L}_{\mathsf{imp}}$

It is easy to turn a fixed-point S_{Ψ^+} into a model of $\mathcal{L}_{\mathsf{imp}}$.

- Let the extension of imp generated by S_{Ψ^+}, in symbols $\mathsf{E}_{S_{\Psi^+}}$, be the set of pairs $\langle \varphi, \psi \rangle$ s.t.

$$\varnothing \Rightarrow \mathsf{imp}(\ulcorner \varphi \urcorner, \ulcorner \psi \urcorner) \in S_{\Psi^+}$$

- Let the anti-extension of imp generated by S_{Ψ^+}, in symbols $\mathsf{A}_{S_{\Psi^+}}$, be the set of pairs $\langle \varphi, \psi \rangle$ s.t.

$$\mathsf{imp}(\ulcorner \varphi \urcorner, \ulcorner \psi \urcorner) \Rightarrow \varnothing \in S_{\Psi^+}$$

- The model of $\mathcal{L}_{\mathsf{imp}}$ associated with I_{Ψ} is $(\mathbb{N}, \mathsf{E}_{\mathsf{I}_{\Psi^+}}, \mathsf{A}_{\mathsf{I}_{\Psi^+}})$.

Outline

Plan of the section

- We now provide (a sketch of) an axiomatic theory, SIT, that axiomatizes adequately the class of models generated by fixed points of Ψ^+.
- We study the proof-theoretical power of SIT by a direct comparison with a theory of symmetric truth (but non-symmetric implication) that has been extensively studied: Partial Kripke-Feferman, PKF (Halbach and Horsten 2006).

The logic of SIT

- The logic of tolerant-strict consequence, TS (Cobreros, Égré, Ripley, and van Rooij 2012).
- Sentences are assigned one of three values, 1, $\frac{1}{2}$, and 0.
- The logical vocabulary is interpreted as in strong Kleene semantics (with imp as the strong Kleene conditional)
- TS-consequence is defined as follows:

$$\Gamma \vDash_{\mathsf{TS}} \Delta :\Leftrightarrow \text{whenever all the sentences in } \Gamma \text{ have value 1 or } \frac{1}{2}, \text{ then at least one sentence in } \Delta \text{ has value 1.}$$

- Reflexivity fails in TS: if φ has value $\frac{1}{2}$, then $\varphi \nvDash_{\mathsf{TS}} \varphi$.

The relationships between TS and K3

- Both TS and K3 (strong Kleene logic) lack logical theorems.
- Moreover, TS-consequence can be expressed by the material conditional of K3. More precisely:

Lemma

The two following claims are equivalent:

(i) If $\Gamma_1 \vDash_{\mathsf{TS}} \Delta_1$, ..., $\Gamma_n \vDash_{\mathsf{TS}} \Delta_n$, then $\Gamma \vDash_{\mathsf{TS}} \Delta$

(ii) $\bigwedge \Gamma_1 \to \bigvee \Delta_1, \ldots, \bigwedge \Gamma_n \to \bigvee \Delta_n \vDash_{\mathsf{K3}} \bigwedge \Gamma \to \bigvee \Delta$

The base syntax theory of SIT

- An initial sequent of the form $\Rightarrow \varphi$, for φ an axiom of Peano Arithmetic.

- The induction rule for the full $\mathcal{L}_{\mathsf{imp}}$:

$$\frac{\Gamma, \varphi(x) \Rightarrow \varphi(x+1), \Delta}{\Gamma, \varphi(0) \Rightarrow \varphi(y), \Delta}$$

The principles for imp: weakening

$$\frac{\Gamma, \mathrm{Sent}(x \underset{\cdot}{\wedge} y) \Rightarrow \mathrm{imp}(x, y), \Delta}{\Gamma \Rightarrow \mathrm{imp}(x \underset{\cdot}{\wedge} v, y \underset{\cdot}{\vee} w), \Delta}\ \text{W-R}$$

$$\frac{\Gamma, \mathrm{Sent}(x \underset{\cdot}{\wedge} y), \mathrm{imp}(x, y) \Rightarrow \Delta}{\Gamma, \mathrm{imp}(x \underset{\cdot}{\vee} v, y \underset{\cdot}{\wedge} w) \Rightarrow \Delta}\ \text{W-L}$$

The principles for imp: conjunction

$$\frac{\Gamma, \mathrm{Sent}(v \dot{\wedge} x) \Rightarrow \mathrm{imp}(v, x), \Delta \qquad \Gamma, \mathrm{Sent}(v \dot{\wedge} y) \Rightarrow \mathrm{imp}(v, y), \Delta}{\Gamma \Rightarrow \mathrm{imp}(v, x \dot{\wedge} y), \Delta} \wedge\text{-R}$$

$$\frac{\Gamma, \mathrm{Sent}(x \dot{\wedge} y \dot{\wedge} v), \mathrm{imp}(x, v), \mathrm{imp}(y, v) \Rightarrow \Delta}{\Gamma, \mathrm{imp}(x \dot{\wedge} y, v) \Rightarrow \Delta} \wedge\text{-L}$$

The principles for imp: implication

$$\frac{\Gamma, \mathsf{Sent}(x \dot{\wedge} y \dot{\wedge} z), \mathsf{imp}(\top, x) \Rightarrow \mathsf{imp}(y, z), \Delta}{\Gamma \Rightarrow \mathsf{imp}(x, \mathsf{imp}(y, z)), \Delta}\ \text{imp-R}$$

$$\frac{\Gamma, \mathsf{Sent}(x) \Rightarrow \mathsf{imp}(\top, x), \Delta \qquad \Gamma, \mathsf{Sent}(y \dot{\wedge} z), \mathsf{imp}(y, z) \Rightarrow \Delta}{\Gamma, \mathsf{imp}(x, \mathsf{imp}(y, z)) \Rightarrow \Delta}\ \text{imp-L}$$

SIT: symmetry

Proposition

Recall the definition of truth via implication. The rules for symmetric truth and implication are derivable rules of SIT*:*

$$\frac{\Gamma, \varphi \Rightarrow \psi, \Delta}{\Gamma \Rightarrow \mathsf{imp}(\ulcorner \varphi \urcorner, \ulcorner \psi \urcorner), \Delta}\ \textsf{imp-I}$$

$$\frac{\Gamma \Rightarrow \varphi, \Delta \qquad \Gamma \Rightarrow \mathsf{imp}(\ulcorner \varphi \urcorner, \ulcorner \psi \urcorner), \Delta}{\Gamma \Rightarrow \psi, \Delta}\ \textsf{imp-E}$$

$$\frac{\Gamma \Rightarrow \varphi, \Delta}{\Gamma \Rightarrow \mathsf{Tr}(\ulcorner \varphi \urcorner), \Delta}\ \textsf{Tr-1} \qquad \frac{\Gamma, \varphi \Rightarrow \Delta}{\Gamma, \mathsf{Tr}(\ulcorner \varphi \urcorner) \Rightarrow \Delta}\ \textsf{Tr-2}$$

Adequacy

Proposition

$(\mathbb{N}, S) \vDash_{\mathsf{TS}} \mathsf{SIT}$ *if and only if* $\Psi^{+}(S) = S$

Proof sketch.

For the left-to-right direction:

- If $(\mathbb{N}, S) \vDash_{\mathsf{TS}} \mathsf{imp}(\ulcorner \varphi \urcorner, \ulcorner \psi \urcorner)$ then, by imp-I, we also have $(\mathbb{N}, S) \vDash_{\mathsf{TS}} \mathsf{imp}(\ulcorner \top \urcorner, \ulcorner \mathsf{imp}(\ulcorner \varphi \urcorner, \ulcorner \psi \urcorner) \urcorner)$.
- If $(\mathbb{N}, S) \vDash_{\mathsf{TS}} \mathsf{imp}(\ulcorner \top \urcorner, \ulcorner \mathsf{imp}(\ulcorner \varphi \urcorner, \ulcorner \psi \urcorner) \urcorner)$ then, by imp-E, we also have $(\mathbb{N}, S) \vDash_{\mathsf{TS}} \mathsf{imp}(\ulcorner \varphi \urcorner, \ulcorner \psi \urcorner)$.

□

Partial Kripke-Feferman

We now recall the essentials of the theory PKF.

It features an axiomatization of PA in strong Kleene logic, and the following truth-theoretical axioms:

PKF1
1. $s^\circ = t^\circ \Leftrightarrow \mathrm{Tr}(s \mathbin{\underset{\cdot}{=}} t)$

PKF2
1. $\mathrm{Sent}(x \mathbin{\underset{\cdot}{\wedge}} y), \mathrm{Tr}(x \mathbin{\underset{\cdot}{\wedge}} y) \Rightarrow \mathrm{Tr}(x) \wedge \mathrm{Tr}(y)$
2. $\mathrm{Sent}(x \mathbin{\underset{\cdot}{\wedge}} y), \mathrm{Tr}(x) \wedge \mathrm{Tr}(y) \Rightarrow \mathrm{Tr}(x \mathbin{\underset{\cdot}{\wedge}} y)$

PKF3
1. $\mathrm{Tr}(t^\circ) \Leftrightarrow \mathrm{Tr}(\underset{\cdot}{\mathrm{T}}\mathrm{r}(t))$

PKF4
1. $\mathrm{Sent}(x), \neg\mathrm{Tr}(x) \Rightarrow \mathrm{Tr}(\underset{\cdot}{\neg}\, x)$
2. $\mathrm{Sent}(x), \mathrm{Tr}(\underset{\cdot}{\neg}\, x) \Rightarrow \neg\mathrm{Tr}(x)$

Implication in PKF

We can formulate PKF in the language of truth and implication, and add appropriate principles for imp:

- We add all the rules of SIT for introducing imp to the left.
- We add the rules for introducing imp to the right with an extra premiss corresponding to LEM. For example:

$$\frac{\Gamma, \mathrm{Sent}(x \dot{\wedge} y \dot{\wedge} z), \mathrm{imp}(\top, x) \Rightarrow \mathrm{imp}(y, z), \Delta}{\Gamma, \mathrm{Tr}(x) \vee \neg \mathrm{Tr}(x) \Rightarrow \mathrm{imp}(x, \mathrm{imp}(y, z)), \Delta} \text{ imp-R}^*$$

We call the resulting theory PKFI.

No loss of power

Proposition

SIT $\vdash \varphi$ *if and only if* PKFI $\vdash \varphi$

Proof sketch.

[$\Rightarrow$] One inductively shows that, if SIT proves $\varphi \Rightarrow \psi$, then either PKFI proves ψ or PKFI proves $\varphi \vee \neg\varphi$. Therefore all SIT-proofs can be safely "mimicked" in PKFI.

[$\Leftarrow$] If PKFI proves φ, then KFint proves $\mathrm{Tr}(\ulcorner\varphi\urcorner)$. But then

$$\mathrm{SIT} \vdash \text{“} \vdash^{k}_{\mathrm{KFint}} \mathrm{Tr}(\ulcorner\varphi\urcorner)\text{”} \rightarrow \mathrm{Tr}_{\omega^k}(\ulcorner\varphi\urcorner)$$

because SIT defines Tarskian truth predicates up to ω^ω. □

An open question

- PKF can be strengthened, adding to it principles for transfinite induction that yield a theory of symmetric truth that proves the same truths of KF (Nicolai 2017).

- The same should apply to SIT. Does it?

Summing up

- SIT is a theory of fully symmetric truth and implication.
- SIT adequately axiomatizes an inductive construction based on the logic TS, that constitutes the substructural dual of Kripke's construction for K3.
- SIT comes at no deductive cost with respect to theories of symmetric truth, since it has the same theorems as PKF.
- What about the second aim of the paper, i.e. attempting a reading for symmetric implication? I_Ψ seems to provide one, by analogy with Kripkean grounded truth: grounded implication (Murzi and Rossi 2017).

Thank you very much!

References I

Beall, J.C. and Murzi, J.
Two Flavours of Curry?s Paradox
The Journal of Philosophy CX, 3, pp.143?65 (2013).

Cobreros, P., Egré, P., Ripley, D., and van Rooij, R.
Tolerant, Classical, Strict
The Journal of Philosophical Logic, 41, pp. 437-385 (2012).

Field, Hartry
Disarming a Paradox of Validity
Notre Dame Journal of Formal Logic 58, 1, pp. 1–19 (2017).

Halbach, V. and Horsten, L.
Axiomatizing Kripke?s Theory of Truth,
The Journal of Symbolic Logic, 71, 2, 677-712 (2006).

Kripke, Saul
Outline of a theory of truth,
Journal of Philosophy 72, 19, pp. 690–716 (1975).

References II

Murzi, J. and Rossi, L.
Naïve Validity,
Synthese, forthcoming (2017).

Nicolai, C. and Rossi, L.
Principles for object-linguistic consequence: from logical to irreflexive
The Journal of Philosophical Logic, forthcoming (2017).

Nicolai, C.
Provably true sentences across axiomatizations of Kripke's theory of truth.
Studia Logica, forthcoming (2017).

Quine, W.V.O.
Three grades of modal involvement
in *The Ways of Paradox*, Brussels, North-Holland Publishing (1953).